9.1

7 (a) let $B = \sup_{i \in \mathbb{N}} \{ |b_i| \}$, Since $\{b_i\}_{i=1}^{\infty}$ bounded $\Rightarrow B < \infty$

$$\sum_{i=m}^{n} |a_i b_i| \leq B \sum_{i=m}^{n} |a_i| \quad \forall\, n > m,\ n, m \in \mathbb{N} \quad ①$$

Since $\{a_i\}_{i=1}^{\infty}$ converges absolutely $\iff \left\{ \sum_{i=1}^{n} |a_i| \right\}_{n=1}^{\infty}$ converges,

$\iff$ (by Cauchy's Convergence Thm):

$\forall\, \varepsilon' > 0$, $\exists\, N \in \mathbb{N}$ s.t. $\forall\, n > m > N$, $n, m, N \in \mathbb{N}$

we have $\left| \sum_{i=1}^{n} |a_i| - \sum_{i=1}^{m} |a_i| \right| < \varepsilon'$ ②

$\forall\, \varepsilon > 0$, take $\varepsilon' = \frac{\varepsilon}{B}$, combining ① & ② we obtain:

$\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$ (actually same N in ②) s.t.

$\forall\, n > m > N$, $n, m, N \in \mathbb{N}$, we have

$$\left| \sum_{i=1}^{n} |a_i b_i| - \sum_{i=1}^{m} |a_i| \right| = \left| \sum_{i=m}^{n} |a_i b_i| \right| \leq |B| \left| \sum_{i=1}^{n} |a_i| \right.$$

$$< |B| \cdot \frac{\varepsilon}{B}$$

$$= \varepsilon$$

(b) Let's take $a_n = (-1)^n \frac{1}{n}$

$b_n = (-1)^n$

Then we check

① $\left\{\sum_{i=1}^{n} a_i\right\}_{n=1}^{\infty}$ is convergent

② $\left\{\sum_{i=1}^{n} a_i b_i\right\}_{n=1}^{\infty}$ diverges

For ① Note that $|a_{2i}+a_{2i-1}| = \left|\frac{1}{2i}-\frac{1}{2i-1}\right| \leq \frac{1}{2i-2}-\frac{1}{2i}$ for $i>1$ (☆)

Then we have $\left|\sum_{i=2m}^{2n} a_i\right| = \left|\sum_{j=m}^{n} a_{2j}+a_{2j+1}\right|$ for $n>m$, $n,m\in\mathbb{N}$

(by ☆) $\leq \sum_{j=m}^{n} \frac{1}{2j}-\frac{1}{2j+2}$

$\leq \frac{1}{2m}+\frac{1}{2n+2} < \frac{1}{m}$

Similarily, we have

$$\left|\sum_{i=2m-1}^{2n} a_i\right| = \left|a_{2m-1}+\sum_{i=2m}^{2n} a_i\right| < \frac{1}{2m-1}+\frac{1}{m} \leq \frac{2}{m}$$

$$\left|\sum_{i=2m-1}^{2n+1} a_i\right| = \left|a_{2m-1}+a_{2n+1}+\sum_{i=2m}^{2n} a_i\right| < \frac{1}{2m-1}+\frac{1}{2n+1}+\frac{1}{m} < \frac{3}{m}$$

$$\left|\sum_{i=2m}^{2n+1} a_i\right| = \left|a_{2n+1}+\sum_{i=2m}^{2n} a_i\right| < \frac{1}{2n+1}+\frac{1}{m} < \frac{2}{m}$$

$\forall\, \varepsilon>0, \exists\, N>\frac{1}{\varepsilon}$, $N\in\mathbb{N}$, s.t. $\forall\, n>m>N$, $m,n\in\mathbb{N}$

We have $\left|\sum_{i=m}^{n} a_i\right| < \frac{1}{N} < \varepsilon$, hence $\left\{\sum_{i=1}^{n} a_i\right\}_{n=1}^{\infty}$ is convergent.

For ②, note that $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} \frac{1}{i}$ is harmonic series.

9.

Since $\{\sum_{i=1}^{n} a_i\}_{n=1}^{\infty}$ converges, for $\forall\ \varepsilon>0$, $\exists\ N\in\mathbb{N}$

s.t. $\forall\ n>m>N$, $n,m\in\mathbb{N}$, we have

$$\sum_{i=m}^{n} a_i < \frac{\varepsilon}{2}$$

Since $\{a_i\}_{i=1}^{\infty}$ monotonely decreasing

$$\Rightarrow (n-m)a_n \le \sum_{i=m}^{n} a_i < \frac{\varepsilon}{2}$$

$$\Rightarrow n a_n < \frac{\varepsilon}{2} + m a_n$$

Since a_i positive, we have $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \lim_{n\to\infty} m a_n = 0 \Rightarrow \exists\ N'\in\mathbb{N}$ s.t. for $\forall\ n>N'$

$$m a_n < \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \forall\ n > \max\{N, N'\}$$

$$0 < n a_n < \varepsilon$$

Since ε is arbitrary $\Rightarrow \limsup_{n} n a_n = 0$

Since $n a_n > 0$, we have $\lim_{n\to\infty} n a_n = 0$

13. (a)

let $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} = \frac{1}{\sqrt{n}(\sqrt{n+1}+\sqrt{n})} > \frac{1}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n+1})} = \frac{1}{2(n+1)}$

$\Rightarrow 2\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} \frac{1}{i+1}$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent since harmonic series is divergent

(b)

let $b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{(\sqrt{n+1}+\sqrt{n})n} < \frac{1}{2n^{\frac{3}{2}}}$

We understand $b'_n = \frac{1}{2(n+1)^{\frac{3}{2}}}$ as the value of function

$f(x) = \frac{1}{2x^{\frac{3}{2}}}$ at $x = n+1$

Since $f(x)$ is decreasing, $\forall n > m > 1$, $n, m \in \mathbb{N}$

$$\Rightarrow \sum_{i=m}^{n} b'_i \leq \sum_{i=m}^{\infty} ((i+1)-i)\, f(i+1) < \int_m^{\infty} f(x)dx = \int_m^{\infty} \frac{1}{2} x^{-\frac{3}{2}} dx$$

$$= -x^{-\frac{1}{2}} \Big|_m^{\infty}$$

$$= \frac{1}{\sqrt{m}}$$

$\Rightarrow \sum_{i=m}^{n} b_n \leq \frac{1}{\sqrt{m-1}}$

for $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, e.g. $N > \frac{1}{\varepsilon^2} + 1$, s.t. for $\forall n > m > N$, $n, m \in \mathbb{N}$, we have $\sum_{i=m}^{n} b_n \leq \frac{1}{\sqrt{N-1}} < \sqrt{\varepsilon^2} < \varepsilon$

$\Rightarrow \sum_{i=1}^{\infty} b_n$ is Convergent.

9.2

(1) (b)

$$\frac{n}{(n+1)(n+2)} \geq \frac{n}{n(n+4)} \quad (\text{for } n \geq 2)$$

$$= \frac{1}{n+4}$$

$$\Rightarrow \sum_{n=1}^{m} b_n > \sum_{n=5}^{m} \frac{1}{n} \quad (\text{for } m > 5)$$

Since RHS is divergent, we have $\left\{\sum_{n=1}^{m} b_n\right\}_{m=1}^{\infty}$ is divergent.

(d)

$$\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \leq \frac{3}{4} \qquad \text{for } n \geq 2$$

by ratio test

we have $\left\{\sum_{n=1}^{m} \frac{n}{2^n}\right\}_{m=1}^{\infty}$ is convergent

(2) (a)

$$\left(\frac{1}{n(n+1)}\right)^{\frac{1}{2}} > \left(\frac{1}{(n+1)^2}\right)^{\frac{1}{2}} = \frac{1}{n+1}$$

$$\Rightarrow \sum_{n=1}^{m} \frac{1}{n(n+1)} > \sum_{n=1}^{m} \frac{1}{n+1}$$

Since harmonic series is divergent,

$\left\{\sum_{n=1}^{m} \frac{1}{n(n+1)}\right\}_{m=1}^{\infty}$ is divergent.

(c) $\dfrac{\dfrac{(n+1)!}{(n+1)^{n+1}}}{\dfrac{n!}{n^n}} = (n+1)\cdot(n+1)^{-n+1} n^n$

$= \left(\dfrac{n}{n+1}\right)^n = \left(1-\dfrac{1}{n+1}\right)^n$

Since $\lim\limits_{n\to\infty} \left(1-\dfrac{1}{n+1}\right)^{n+1} = \dfrac{1}{e}$

$\Rightarrow \lim\limits_{n\to\infty} \left(1-\dfrac{1}{n+1}\right)^n = \lim\limits_{n\to\infty} \left(1-\dfrac{1}{n+1}\right)^{n+1} \cdot \dfrac{1}{1-\frac{1}{n+1}} = \dfrac{1}{e}$

$\Rightarrow$ there $\exists\, K\in\mathbb{N}$, s.t. $\forall\, n>k$, $n\in\mathbb{N}$,

$\left(1-\dfrac{1}{n+1}\right)^n < \dfrac{1}{2}$

by Ratio Test

we have $\left\{\sum\limits_{n=1}^{m} \dfrac{n!}{n^n}\right\}_{m=1}^{\infty}$ is convergent

(3) (e) Consider function $f(x)=\dfrac{1}{x\ln x}$ which is >0 & decreasing on $[e,+\infty)$

$\lim\limits_{b\to\infty} \int_e^b \dfrac{1}{x\ln x}\,dx = \lim\limits_{b\to\infty} \int_e^b \dfrac{1}{\ln x}\, d\ln x \overset{t=\ln x}{=} \lim\limits_{b\to\infty} \int_1^{\ln b} \dfrac{1}{t}\,dt$

$= \lim\limits_{b\to\infty} \ln\ln b$

is divergent, for $n\geq 3>e$

by Integral test,

we have $\left\{\sum\limits_{n=3}^{m} \dfrac{1}{n\ln n}\right\}_{m=3}^{\infty}$ is divergent

(f) $f(x) = \dfrac{1}{x\ln x(\ln\ln x)^2}$ is >0 & decreasing on $[e^e, +\infty)$

$$\int_{e^e}^{b} f(x)dx = \int_{e^e}^{b} \frac{1}{\ln x(\ln\ln x)^2} d\ln x \overset{t=\ln x}{=\!=} \int_{e}^{\ln b} \frac{1}{t(\ln t)^2} dt$$

$$\overset{s=\ln t}{=\!=} \int_{1}^{\ln\ln b} \frac{1}{s^2} ds$$

$$= 1 - \frac{1}{\ln\ln b}$$

$$\Rightarrow \lim_{b\to\infty} \int_{e^e}^{b} f(x)dx = \lim_{b\to\infty} 1 - \frac{1}{\ln\ln b} = 1$$

By Integral test, for $n \geq 27 > e^e$

we have $\left\{ \sum_{n=27}^{m} \dfrac{1}{n\ln n(\ln\ln n)^2} \right\}_{m=27}^{\infty}$ is convergent